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THE PRODUCT OF NONPLANAR COMPLEXES DOES NOT IMBED IN 4-SPACE

BY

BRIAN R. UMMEL

ABSTRACT. We prove that if K_1 and K_2 are nonplanar simplicial complexes, then $K_1 \times K_2$ does not imbed in $\mathbf{R}^4$.

In this paper a proof is given of the following theorem:

THEOREM P. *If K_1 and K_2 are finite simplicial complexes neither of which is homeomorphic to a subset of the euclidean plane $\mathbf{R}^2$, then their cartesian product $K_1 \times K_2$ is not homeomorphic to any subset of euclidean 4-space $\mathbf{R}^4$.*

This result answers a question originally posed by Professor Karl Menger in [5]. I wish to thank Professor Joseph Zaks for showing me this problem.

1. Preliminaries. We say a space X *imbeds* in euclidean n-space $\mathbf{R}^n$ if there is an imbedding (i.e., homeomorphism into) $f: X \to \mathbf{R}^n$. If X imbeds in $\mathbf{R}^2$ we say that X is *planar*. For a proof of the following see [4]:

PROPOSITION 1.1. *If K is a finite nonplanar simplicial complex then K contains a subspace homeomorphic to one of the following spaces:*

a. K_5^1, *the complete graph on 5 vertices (or, if you prefer, the 1-skeleton of a 4-simplex);*

b. $K_{3,3}^1$, *the join of 2 3-point sets;*

c. S^2, *the 2-sphere; or*

d. $Q^2 = \{x \in \mathbf{R}^3: x_3 = 0$ *and* $x_1^2 + x_2^2 \leq 1$ *or* $x_1 = x_2 = 0$ *and* $0 \leq x_3 \leq 1\}$.

We will henceforth assume that the complexes K_1 and K_2 of Theorem P are chosen from the list of Proposition 1.1, and that we have chosen imbeddings

$$f_1: K_1 \to \mathbf{R}^3 \text{ and } f_2: K_2 \to \mathbf{R}^3.$$

To clarify notations we recall some standard definitions. Let $\pi = \{1, \tau\}$ be the multiplicative group of order 2. A *π-space X* is a Hausdorff space together with a fixed point free involution $\tau: X \to X$; this involution defines a free

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π-action on X and we denote the orbit space of this action by X/π. The natural projection p: $X \to X/\pi$ is then a 2-fold covering. If X and Y are π-spaces then a map f: $X \to Y$ is *π-equivariant* if $f \cdot \tau = \tau \cdot f$. Homotopies are equivariant if they are π-equivariant at each stage. If K is a Hausdorff space, the deleted product of K is

$$D_2K = \{(x, y) \in K \times K: x \neq y\}.$$

Using the action $\tau(x, y) = (y, x)$, D_2K is a π-space and we denote the orbit space by Σ_2K. Let $S^\infty = \text{proj lim } S^n$ under the natural inclusions and τ: $S^\infty \to S^\infty$ be the limit of the antipodal maps; set $P^\infty = S^\infty/\tau$. If K is paracompact there is a π-equivariant map $\hat{k}$: $D_2K \to S^\infty$ and any two such maps are π-equivariantly homotopic (cf. [2]). Using the induced map k: $\Sigma_2K \to P^\infty$ and singular cohomology with Z_2 coefficients we define the *nth* mod-2 *imbedding* class of K by

$$\Phi_2^n(K) = K^*(w^n) \in H^n(\Sigma_2K; Z_2)$$

where w^n is the nonzero element of $H^n(P^\infty; Z_2)$, $n \geqslant 0$. The following is an immediate consequence of the definition.

Proposition 1.2. a. *If K and L are paracompact, f: $K \to L$ is an imbedding, and F: $\Sigma_2K \to \Sigma_2L$ is the induced map, then $F^*(\Phi_2^n(L)) = \Phi_2^n(K)$.* b. *For $n > 0$, $D_2\mathbf{R}^n$ is π-equivariantly homotopy equivalent to S^{n-1} (with antipodal action); thus $\Phi_2^i(\mathbf{R}^n) \neq 0$ iff $0 \leqslant i \leqslant n - 1$.*

Thus $\Phi_2^n(K) = 0$ is a necessary condition for a paracompact space to imbed in $\mathbf{R}^n$. In §3 we prove Theorem P by showing that $\Phi_2^4(K_1 \times K_2) \neq 0$. The information we need about the deleted products of K_1 and K_2 is summarized in the following:

Proposition 1.3. *If K is one of the four complexes K_5^1, $K_{3,3}^1$, S^2 or Q^2 of Proposition 1.1, then*

a. $\Phi_2^2(K) \neq 0$;

b. *D_2K is π-equivariantly homotopy equivalent to a closed 2-manifold of genus g, where $g = 6$ if $K = K_5^1$, $g = 4$ if $K = K_{3,3}^1$ and $g = 0$ if $K = S^2$ or Q^2;*

c. *if f: $K \to \mathbf{R}^3$ is an imbedding and $\hat{F}$: $D_2K \to D_2\mathbf{R}^3$ is the induced map, then $\hat{F}^*$: $H^2(D_2\mathbf{R}^3) \to H^2(D_2K)$ is an isomorphism.*

Proof. For a and b see [7] and [8]. For c we have the following commutative diagram whose rows are exact Gysin sequences (where we interpret a 2-fold covering as a 0-sphere bundle cf. [6]):

$$\begin{array}{ccccccc}
\to & H^2(\Sigma_2\mathbf{R}^3) & \overset{p'^*}{\to} & H^2(D_2\mathbf{R}^3) & \overset{\rho'}{\to} & H^2(\Sigma_2\mathbf{R}^3) & \to 0 \\
 & \downarrow F^* & & \downarrow \hat{F}^* & & \downarrow F^* & \\
\to & H^2(\Sigma_2K) & \overset{p^*}{\to} & H^2(D_2K) & \overset{\rho}{\to} & H^2(\Sigma_2K) & \to 0
\end{array}$$

where F: $\Sigma_2 K \to \Sigma_2 \mathbf{R}^3$ is the map induced by $\hat{F}$. All six groups in the diagram are isomorphic to Z_2 and so ρ and ρ' are isomorphisms. Thus $\hat{F}^*$ is an isomorphism.

Using $[g_1, \ldots, g_n]$ to denote the Z_2-module with basis $\{g_1, \ldots, g_n\}$ or the zero module if $n = 0$, we can write

$$H^0(K_1) = [\omega^0], \qquad H^0(K_2) = [\mu^0],$$
$$H^1(K_1) = [\omega^1, \ldots, \omega_\eta^1], \quad H^1(K_2) = [\mu_1^1, \ldots, \mu_\sigma^1],$$

where η {or σ} is 0, 0, 4, or 6 depending upon whether K_1 {or K_2} is S^2, Q^2, $K_{3,3}^1$, or K_5^1. Here the superscripts denote dimension rather than exponents. If K_1 $\{K_2\}$ is S^2 or Q^2 we denote $H^2(K_1) = [\omega^2]$ $\{H^2(K_2) = [\mu^2]\}$; otherwise $H^2(K_1) = 0$ $\{H^2(K_2) = 0\}$. We also need to assume that if $\eta \neq 0$ $\{\sigma \neq 0\}$ then the above basis for $H^1(K_1)$ $\{H^1(K_2)\}$ is dual to a basis which satisfies the following:

LEMMA 1.4. *If K is a finite 1-dimensional simplicial complex and i: $D_2K \to K \times K$ is the inclusion map, then there is a basis $\{\beta_1, \ldots, \beta_m\}$ for $H_1(K)$ such that if $\beta \in H_2(D_2K)$ then $i_*(\beta) = \Sigma c_{ij}(\beta_i \times \beta_j)$ where $c_{ij} \in Z_2$, $c_{ii} = 0$ for $i = 1, \ldots, m$ and "$\times$" denotes cross product.*

PROOF. Let $D_2^P(K) = \{(x_1, x_2) \in K \times K\colon c(x_1) \cap c(x_2) = \varnothing\}$ where $c(x_j)$ is the smallest closed simplex of K containing x_j. Then, by [9], $D_2^P(K)$ is a strong π-equivariant deformation retract of D_2K, and so we can use the inclusion j: $D_2^PK \to K \times K$ instead of i. Let $\{\sigma_1, \ldots, \sigma_{m+n}\}$ be the 1-simplices of K numbered so that $\{\sigma_{m+1}, \ldots, \sigma_{m+n}\}$ form a maximal tree Γ of K. We also use σ_i to denote the linear singular 1-simplex whose image is σ_i; there is no orientation problem since we are using Z_2 coefficients. For $i = 1, \ldots, m$ set $\beta_i = [\sigma_i + \lambda_i] \in H_1(K_1)$ where λ_i is a sum of simplices of Γ. For $i > m$ we set $\lambda_i = \sigma_i$. Suppose $\beta \in H_2(D_2^PK)$. Then $\beta = [\Sigma k_{ij}(\sigma_i \times \sigma_j)]$ where $k_{ij} \in Z_2$ and $k_{ii} = 0$ for all i. We have

$$\begin{aligned}\sigma_i \times \sigma_j &= (\sigma_i + \lambda_i - \lambda_i) \times (\sigma_j + \lambda_j - \lambda_j)\\ &= (\sigma_i \times \lambda_i) \times (\sigma_j \times \lambda_j) - \lambda_i \times \sigma_j - \sigma_i \times \lambda + \lambda_i + \lambda_j.\end{aligned}$$

So if $i \neq j$, $\sigma_i \times \sigma_j = (\sigma_i + \lambda_j) + \gamma_{ij}$ where γ_{ij} is a 2-chain of $X \times \Gamma \cup \Gamma \times X$. So $\beta = [\Sigma k_{ij}(\sigma_i + \lambda_i) \times (\sigma_j + \lambda_j) + \gamma]$ where γ is a 2-chain of $\Gamma \times X \cup X \times \Gamma$. In fact γ is a 2-cycle of $\Gamma \times X \cup X \times \Gamma$ and since $H_2(\Gamma \times X \cup X \times \Gamma) = 0$, $j_*(\gamma) = 0$. Thus $j^*(\beta) = \Sigma k_{ij}(\beta_i \times \beta_j)$. Thus $\{\beta_1, \ldots, \beta_n\}$ is the desired basis. This proves Lemma 1.4.

LEMMA 1.5. *If $K = K_5^1$, $K_{3,3}^1$, S^2 or Q^2, then the inclusion map J: $D_2K \to K \times K$ induces an isomorphism j^*: $H^1(K \times K) \to H^1(D_2K)$.*

PROOF. For $K = S^2$ or Q^2 both groups are trivial. If $K = K_5^1$ or $K_{3,3}^1$ then by [8] $D_2(CK)$ is π-equivariantly homotopy equivalent to S^3. So in the exact sequence (cf. [1])

$$\to H^n(D_2CK) \to H^n(K) \oplus H^n(K) \xrightarrow{\alpha} H^n(D_2K) \to H^{n+1}(D_2CK) \to$$

where $\alpha(u, v) = q_1^*(u) + q_2^*(v)$, q_i: $D_2K \to K$ given by $q_i(x_1, x_2) = x_i$, we have α is an isomorphism if $n = 1$. Using this and the Künneth Theorem proves Lemma 1.5.

Using Lemma 1.5, Proposition 1.5 and the above bases we have:

$$H^0(D_2K_1) = [\omega^0 \times \omega^0], \quad H^1(D_2K_1) = [\omega^0 \times \omega_i^1, \omega_i^1 \times \omega^0; i = 1, \ldots, \eta],$$
$$H^2(D_2K_1) = [\Omega^2],$$

and

$$H^0(D_2K_2) = [\mu^0 \times \mu^0], \quad H^1(D_2K_2) = [\mu^0 \times \mu_i^1, \mu_i^1 \times \mu^0; i = 1, \ldots, \sigma],$$
$$H^2(D_2K_2) = [\Lambda^2]$$

where "$\times$" denotes cross product followed by restriction.

For Hausdorff spaces K and L we define

$$\hat{J}_0(K, L) = D_2K \times D_2L, \quad \hat{J}_1(K, L) = D_2K \times (L \times L),$$
$$\hat{J}_2(K, L) = (K \times K) \times D_2L.$$

Using $\tau(x_1, x_2, y_1, y_2) = (x_2, x_1, y_2, y_1)$, $\hat{J}_k(K, L)$ becomes a π-space and we denote the quotient spaces by $J_k(K, L)$ for $k = 0, 1, 2$.

Lemma 1.6. *If K and L are Hausdorff spaces then $D_2(K \times L)$ is π-equivariantly homeomorphic to $\hat{J}_1(K, L) \cup \hat{J}_2(K, L)$. Moreover, $\{J_1(K, L), J_2(K, L)\}$ is an excisive couple and $J_1(K, L) \cap J_2(K, L) = J_0(K, L)$.*

Proof. Clearly $\hat{\varphi}$: $D_2(K \times L) \to \hat{J}_1(K, L) \cup \hat{J}_2(K, L)$ defined by

$$\hat{\varphi}(x_1, y_1, x_2, y_2) = (x_1, x_2, y_1, y_2)$$

is a π-equivariant homeomorphism. Since $J_1(K, L)$ and $J_2(K, L)$ are open in their union, the couple is excisive.

For $k = 0$, 1, and 2 let

$$\hat{J}_k = \hat{J}_k(K_1, K_2), \quad \hat{J}_k' = \hat{J}_k(\mathbf{R}^3; \mathbf{R}^3),$$
$$J_k = J_k(K_1, K_2), \quad J_k' = J_k(\mathbf{R}^3; \mathbf{R}^3),$$

and $\hat{F}_k$: $\hat{J}_k \to \hat{J}_k$, F_k: $J_k \to J_k'$ denote the maps induced by the imbeddings f_j: $K_j \to \mathbf{R}^3, j = 1, 2$. We also have maps

$$\hat{F}: D_2(K_1 \times K_2) \to D_2(\mathbf{R}^3 \times \mathbf{R}^3)$$

and

$$F: \Sigma_2(K_1 \times K_2) \to \Sigma_2(\mathbf{R}^3 \times \mathbf{R}^3).$$

Finally let $\hat{\imath}_k$: $\hat{J}_0 \to \hat{J}_k$ and i_k: $J_0 \to J_k$ be the inclusions for $k = 1, 2$ and p_k: $\hat{J}_k \to J_k$, p'_k: $\hat{J}'_k \to J'_k$ be the natural projections for $j = 0, 1, 2$.

LEMMA 1.7. *F_0^*: $H^4(J'_0) \to H^4(J_0)$ is an isomorphism.*

PROOF. $\hat{J}'_0$ and $\hat{J}_0$ are π-equivariantly homotopy equivalent to closed 4-manifolds; hence J_0 and J'_0 are homotopy equivalent to closed 4-manifolds. Thus in the commutative diagram

$$\begin{array}{ccc} H^4(\hat{J}'_0) & \xrightarrow{\rho'} & H^4(J_0) \\ \hat{F}_0^* \downarrow & & \downarrow F_0^* \\ H^4(\hat{J}_0) & \xrightarrow{\rho} & H^4(J_0) \end{array}$$

where ρ' and ρ are from the appropriate Gysin sequences, and Proposition 1.3.c, F_0^* is an isomorphism. This proves Lemma 1.7.

2. The spectral sequence of a double covering. The proof of Theorem P requires the following in which we use the notation of §1.

LEMMA 2.1. Ker$[p_1^*$: $H^4(J_1) \to H^4(\hat{J}_1)] \subseteq$ Im$[i_1^*$: $H^4(J_1) \to H^4(J_0)]$.

The proof of Lemma 2.1 requires using the cohomology spectral sequence of a covering (cf. [3]) specialized to the case of a double covering which allows explicit identification of the E_1-term and the E_1 differential operators. The properties of this spectral sequence are summarized in the following:

PROPOSITION 2.2. *If X is a π-space, there is a natural first quadrant E_1-spectral sequence $\{E_r^{p,q}(X), d_r^{p,q}\}_{r=1}^{\infty}$ convergent to $H^*(X/\pi, Z_2)$ with the following properties*:

a. $E_1^{p,q}(X) = H^q(X; Z_2)$,

b. *$d_1^{p,q}$: $E_1^{p,q}(X) \to E_1^{p+1,q}(X)$ is given by $d_1^{p,q}(\alpha) = \alpha + \tau\alpha$ where τ: $H^1(H; Z_2) = H^1(X, Z_2)$ is the homomorphism induced by the involution τ: $X \to X$.*

c. *For each n there is a natural decreasing filtration $\{F_pH^n(X/n)\}_{p=0}^{\infty}$ of $H^n(X_\pi, Z_2)$ such that $F_0H^n(X/\pi) = H^n(X/\pi; Z_2)$, $F_{n+1}H^n(X/\pi) = 0$ and for each $p \geqslant 0$ there is a natural short exact sequence*

$$0 \to F_{p+1}H_n(X/\pi) \to F_pH_n(X/\pi) \xrightarrow{q} E_\infty^{p,n-p}(X) \to 0.$$

d. *The projection induced map p^*: $H^n(X/\pi; Z_2) \to H^n(X; Z_2)$ is the composition*:

$$H^n(X/\pi) = F_0H^n(X/\pi) \xrightarrow{q} E_\infty^{0,n}(X) \subseteq E_1^{0,n}(X) = H'(X).$$

For the remainder of this section and the next we assume all coefficients to be Z_2 and suppress this in the notation.

LEMMA 2.3. *If* $p \geqslant 1$ *then* $d_3^{p,2}\colon E_3^{p,2}(D_2K_1) \to E_3^{p+3,0}(D_2K_1)$ *is an isomorphism with* $E_3^{p,2}(D_2K_1) = [\Omega^2]$.

PROOF. Using the calculations of §1 and Proposition 2.2.b,

$$E^{p,0}(D_2K_1) = [\omega^0 \times \omega^0], \quad p \geqslant 0, \quad E_2^{p,1}(D_2K_1) = 0, \quad p \geqslant 1,$$
$$E_2^{p,2}(D_2K_1) = [\Omega^2], \quad p \geqslant 0.$$

Thus $E_3^{p,0}(D_2K_1) = E_2^{p,0}(D_2K_1) = [\omega^0 \times \omega^0]$ if $p \geqslant 3$ and $E_3^{p,2}(D_2K_1) = E_2^{p,2}(D_2K_1) = [\Omega^2]$ if $p \geqslant 0$. Since $H^n(\Sigma_2K_1) = 0$ if $n > 2$,

$$d_3^{p,2}\colon E_3^{p,2}(D_2K_1) \to E_3^{p+3,0}(D_2K_1)$$

must be an isomorphism. This proves Lemma 2.3.

LEMMA 2.4. $E_\infty^{p,4-p}(\hat{J}_0) = 0$ *if* $p \geqslant 3$.

PROOF. Using the calculations of §1 and the Künneth formula,

$$H^0(\hat{J}_0) = [\omega^0 \times \omega^0] \otimes [\mu^0 \times \mu^0],$$
$$H^1(\hat{J}_0) = [\omega^0 \times \omega^0] \otimes [\mu_i^1 \times \mu^0, \mu^0 \times \mu_i^1; i = 1, \dots, \sigma]$$
$$\oplus [\omega_i^1 \times \omega^0, \omega^0 \times \omega_i^1; i = 1, \dots, \eta] \otimes [\mu^0 \times \mu^0],$$

where η and σ are the ranks of $H^1(K_1)$ and $H^1(K_2)$ respectively. Using the diagonal π-action and Proposition 2.2.b

$$E_2^{4,0}(\hat{J}_0) = [\omega^0 \times \omega^0] \otimes [\mu^0 \times \mu^0], \quad E_2^{p,1}(\hat{J}_0) = 0 \quad \text{if } p > 0.$$

Since $E_2^{2,1}(\hat{J}_0) = 0$, $E_3^{4,0}(\hat{J}_0) = E_2^{4,0}(\hat{J}_0)$ and

$$\pi_1^*\colon E_3^{4,0}(D_2K_1) \to E_3^{4,0}(\hat{J}_0)$$

is an isomorphism where $\pi_1\colon \hat{J}_0 \to D_2K_1$ is the projection. From the commutative diagram

$$\begin{array}{ccc} E_3^{1,2}(D_2K_1) & \xrightarrow{\pi_1^*} & E_3^{1,2}(\hat{J}_0) \\ d_3^{1,2}\downarrow & & \downarrow d_3^{1,2} \\ E_3^{4,0}(D_2K_1) & \xrightarrow{\pi_1^*} & E_3^{4,0}(\hat{J}_0) \end{array}$$

and Lemma 2.3, $d_3^{1,2}\colon E_3^{1,2}(\hat{J}_0) \to E_3^{4,0}(\hat{J}_0)$ is surjective. Thus $E_\infty^{4,0}(\hat{J}_0) = E_4^{4,0}(\hat{J}_0) = E_4^{4,0}(\hat{J}_0) = 0$. From above, $E_\infty^{3,1}(\hat{J}_0) = 0$. This proves Lemma 2.4.

LEMMA 2.5. $E_\infty^{1,3}(\hat{J}_1) = 0$ *and* $\hat{\imath}^*\colon E_\infty^{2,2}(\hat{J}_0)$ *is the zero homomorphism.*

PROOF. We first compute $E_2^{p,q}(\hat{J}_1)$ in 3-cases. If $K_2 = K_5^1$ or $K_{3,3}^1$, then

$$H^1(\hat{J}_1) = [\omega^0 \times \omega^0] \otimes [\mu^0 \times \mu_i^1, \mu_i^1 \times \mu_0; i = 1, \ldots, \sigma]$$
$$\oplus [\omega^0 \times \omega_i^1, \omega_i^1 \times \omega^0; i = 1, \ldots, \eta] \otimes [\mu^0 \times \mu^0],$$
$$H^2(\hat{J}_1) = [\omega^0 \times \omega^0] \otimes [\mu_i^1 \times \mu_j^1; i,j = 1, \ldots, \sigma]$$
$$\oplus [\omega^0 \times \omega_i^1, \omega_i^1 \times \omega^0; i = 1, \ldots, \eta]$$
$$\otimes [\mu^0 \times \mu_i^1, \mu_i^1 \times \mu^0; i = 1, \ldots, \sigma]$$
$$\oplus [\Omega^2] \otimes [\mu^0 \times \mu^0].$$

Using the diagonal π-action on $\hat{J}_1$ and Proposition 2.2

$$E_2^{p,1}(\hat{J}_1) = 0 \quad \text{if } p > 0,$$
$$E_2^{2,2}(\hat{J}_1) = [\omega^0 \times \omega^0] \otimes [\mu_i^1 \times \mu_i^1; i = 1, \ldots, \sigma] \oplus [\Omega^2] \otimes [\mu^0 \times \mu^0],$$
$$E_2^{1,3}(\hat{J}_1) = 0.$$

If $K_2 = S^2$ and we set $H_2(K_2) = [\mu^2]$ then

$$H^1(\hat{J}_1) = [\omega^0 \times \omega_i^1, \omega_i^1 \times \omega^0; i = 1, \ldots, \eta] \otimes [\mu^0 \times \mu^0],$$
$$H^2(\hat{J}_1) = [\omega^0 \times \omega^0] \otimes [\mu^0 \times \mu^2, \mu^2 \times \mu^0] \oplus [\Omega^2] \otimes [\mu^0 \times \mu^0].$$

Therefore

$$E_2^{p,1}(\hat{J}_1) = 0 \text{ if } p > 0, \quad E_2^{2,2}(\hat{J}_1) = [\Omega^2 \times \mu^0 \times \mu^0],$$
$$E_2^{1,3}(\hat{J}_1) = 0.$$

Finally if $K_2 = Q_2$, then

$$E_2^{p,1}(\hat{J}_1) = 0 \text{ if } p > 0, \quad E_2^{2,2}(\hat{J}_1) = H^2(\hat{J}_1) = [\Omega^2 \times \mu^0 \times \mu^0],$$
$$E_2^{1,3}(\hat{J}_1) = H^3(\hat{J}_1) = 0.$$

In all cases $E_2^{p,0}(\hat{J}_1) = [\omega^0 \times \omega^0 \times \mu^0 \times \mu^0]$. Since $\pi_1^*\colon H^1(D_2K_1) \to H^2(\hat{J}_1)$ is an isomorphism, where $\pi_1\colon \hat{J}_1 \to D_2K_1$ is the projection, and $E_2^{p,1}(\hat{J}_1) = 0$ if $p > 0$, we have

$$\pi_1^*\colon E_3^{p,0}(D_2K_1) \to E_3^{p,0}(\hat{J}_1)$$

is an isomorphism for $p > 2$. Consider the commutative diagram

$$\begin{array}{ccc} E_3^{2,2}(D_2K_1) & \xrightarrow{\pi_1^*} & E_3^{2,2}(\hat{J}_1) \\ d_3^{2,2}\downarrow & & \downarrow d_3^{2,2} \\ E_3^{5,0}(D_2K_1) & \xrightarrow{\pi_1^*} & E_3^{5,0}(\hat{J}_1) \end{array}$$

Using Lemma 2.3, and $\pi_1^*([\Omega^2]) = [\Omega^2 \times \omega^0 \times \omega^0]$ we have

$$d_3^{2,2}([\Omega^2 \times \omega^0 \times \omega^0]) \neq 0.$$

Therefore

$$E_4^{2,2}(\hat{J}_1) = \begin{cases} [\omega^0 \times \omega^0 \times \mu_i^1 \times \mu_i^1; i = 1, \ldots, \sigma] & \text{if } K_2 = K_5' \text{ or } K_{3,3}', \\ 0 \quad \text{if } K_2 = S^2 \text{ or } Q^2. \end{cases}$$

To show that $\hat{\imath}^*([\omega^0 \times \omega^0 \times \mu_i^1 \times \mu_i^1]) = 0$, let $\{\alpha_1^1, \ldots, \alpha_\sigma^1\}$ be the basis of $H_1(K_2)$ dual to $\{\mu_1^1, \ldots, \mu_\sigma^1\}$ as in Lemma 1.4 and let α^2 denote the nonzero element of $H_2(D_2K_2)$. If j: $D_2K_2 \to K_2 \times K_2$ is the inclusion, then by Lemma 1.4

$$\langle j^*(\mu_i^1 \times \mu_i^1), \alpha^2 \rangle = \langle \mu_i^1 \times \mu_i^1, j_*(\alpha^2) \rangle = \left\langle \mu_i^1 \times \mu_i^1, \sum_{j \neq k} c_{jk}(\alpha_j^1 \times \alpha_k^1) \right\rangle$$

$$= \sum_{j \neq k} c_{jk} \langle \mu_i^1, \alpha_j^1 \rangle \langle \mu_i^1, \alpha_k^1 \rangle = \sum_{j \neq k} c_{jk}\delta_{jk} = 0.$$

Thus $j^*(\mu_i^1 \times \mu_i^1) = 0$. So $\hat{\imath}^*([\omega^0 \times \omega^0 \times \mu_i^1 \times \mu_i^1]) = 0$. This proves Lemma 2.5.

PROOF OF LEMMA 2.1. By Proposition 2.2.d, if p_1: $\hat{J}_1 \to J_1$ is the natural projection then p_1^*: $H^4(J_1) \to H^4(\hat{J}_1)$ is the composition

$$H^4(J_1) = F^0H^4(J') \xrightarrow{q} E_\infty^{0,4}(\hat{J}_1) \subseteq E_1^{0,4}(\hat{J}_1) = H^4(J_1).$$

So using Proposition 2.2.c we have

$$\ker[p_1^*: H^4(J_1) \to H^4(\hat{J}_1)]$$
$$= \ker[q: F^0H^4(J_1) \to E_\alpha^{0,4}(\hat{J}_1)] = F_1H^4(J_1).$$

By Lemma 2.5, $E_\infty^{1,3}(\hat{J}_1) = 0$; thus $F_2H^4(J_1) = F_1H^4(J_1)$. Thus $i_1^*|(\ker p_1^*) = 0$ if and only if $i_1^*|F_2H^4(J_1) = 0$. Now consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \to & F_3H^4(J_1) & \to & F_2H^4(J_1) & \to & E_\infty^{2,2}(\hat{J}_1) & \to 0 \\ & \downarrow i_1^* & & \downarrow i_1^* & & \downarrow \hat{\imath}_1^* & \\ 0 \to & F_3H^4(J_0) & \to & F_2H^4(J_0) & \to & E_\infty^{2,2}(\hat{J}_0) & \to 0 \end{array}$$

By Lemma 2.4, $E_\infty^{4,1}(\hat{J}_0) \cong 0$; thus $F_3H^4(J_0) = 0$ and so the projection

$$F_2H^4(J_0) \to E_\infty^{2,2}(\hat{J}_0)$$

is an isomorphism. But by Lemma 2.5, $\hat{\imath}_1^*$: $E_\infty^{2,2}(\hat{J}_1) \to E_\infty^{2,2}(\hat{J}_0)$ is zero. Thus i_1^*: $F_2H^4(J_1) \to F_2H^4(J_0)$ is zero. This proves Lemma 2.1.

3. Proof of Theorem P. We use the notation of §1 including the assumptions that K_1 and K_2 are complexes selected from the list in Proposition 1.1 and that all coefficients are Z_2. We will prove that $\Phi_2^4(K_1 \times K_2) \neq 0$.

Consider first the following commutative diagram in which the rows are the exact Mayer Vietoris sequences given by Lemma 1.6:

$$\begin{array}{ccccccc} H^3(J_1') \oplus H^3(J_2') \to & H^3(J_0') & \xrightarrow{\delta'^*} & H^4(\Sigma_2(\mathbf{R}^3 \times \mathbf{R}^3)) & \to H^4(J_1') \oplus H^4(J_2') \\ & F_0^* \downarrow & & F^* \downarrow & \\ H^3(J_1) \oplus H^3(J_2) \xrightarrow{i_1^* + i_2^*} & H^3(J_0) & \xrightarrow{\delta^*} & H^4(\Sigma_2(K_1 \times K_2)) & \end{array}$$

Since $H^j(J_1') \cong H^j(J_2') \cong H^j(\mathbf{R}P^2) = 0$ for $j > 2$, δ'^* is an isomorphism. Since $\Phi_2^4(K_1 \times K_2) \neq 0$ if $F^* \neq 0$, we have $\Phi_2^4(K_1 \times K_2) \neq 0$ if $\operatorname{Im}(F_0^*) \not\subset \operatorname{Im}(i_1^* + i_2^*)$.

Now consider the following commutative diagram in which the rows are exact Gysin sequences or sums of Gysin sequences:

$$\begin{array}{ccccccc} H^3(J_0') & \xrightarrow{\Delta'^*} & H^4(J_0') & \xrightarrow{P_0'^*} & H^4(\hat{J}_0') & \xrightarrow{\rho'} & H^4(J_0') \\ \downarrow F_0^* & & \downarrow F_0^* & & \downarrow \hat{F}_0^* & & \downarrow F_0^* \\ H^3(J_0) & \xrightarrow{\Delta^*} & H^4(J_0) & \xrightarrow{P_0^*} & H^4(\hat{J}_0) & \xrightarrow{\rho} & H^4(J_0) \\ \downarrow i_1^* + i_2^* & & \nwarrow i_1^* + i_2^* & & \nwarrow \hat{i}_1^* + \hat{i}_2^* & & \\ H^3(J_1) \oplus H^3(J_2) & \xrightarrow{\Delta_1^* + \Delta_2^*} & H^4(J_1) \oplus H^4(J_2) & \xrightarrow{P_1 \oplus P_2} & H^4(\hat{J}_1) \oplus H^4(\hat{J}_2) & & \end{array}$$

Since $\hat{J}_0'$ is π-equivariantly homotopy equivalent to $S^2 \times S^2$, $H^4(\hat{J}_0') \cong H^4(J_0') \cong H^3(J_0') \cong Z_2$ and Δ'^* and ρ' are isomorphisms. By Proposition 1.3, $H^4(\hat{J}_0) \cong H^4(J_0) \cong Z_2$ and by exactness ρ is an isomorphism. By Lemma 1.7, $F_0^*\colon H^4(J_0') \to H^4(J_0)$ is an isomorphism. Let α be the nonzero element of $H^3(J_0')$. We need to show there do not exist elements $\alpha_1 \in H^3(J_1)$ and $\alpha_2 \in H^3(J_2)$ such that $i_1^*(\alpha_1) + i_2^*(\alpha_2) = F_0^*(\alpha)$. Since $F_0^*(\Delta'^*(\alpha)) \neq 0$, it suffices to show that

$$i_1^*(\Delta_1^*(\alpha_1)) + i_2^*(\Delta_2^*(\alpha_2)) = 0.$$

By exactness and symmetry this follows if we prove

$$\ker\left[p_1^*\colon H^4(J_1) \to H^4(\hat{J}_1)\right] \subseteq \ker\left[i_1^*\colon H^4(J_1) \to H^4(J_0)\right].$$

This is exactly what was proven in Lemma 2.1. Thus the proof of Theorem P is complete.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MILWAUKEE, WISCONSIN 53201

Current address: Applied Mathematics Group, Energy Technology Applications Division, Boeing Computer Services Company, P.O. Box 24346, Seattle, Washington 98124